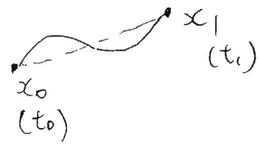
Lagrangean の導出.

$I(x) = \int_{t_0}^{t_1} F(x, \dot{x})\, dt$ を考える. $I(x)$: 作用 (action)

$x \to x + \delta x$ としたとき $\delta I = I(x+\delta x) - I(x) = 0$ となるようにする.

($t = t_0, t = t_1$ では $\delta x = 0$)



$$\delta I = \int_{t_0}^{t_1} \left\{ F(x+\delta x, \dot{x} + \delta \dot{x}) - F(x, \dot{x}) \right\} dt$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x} \delta x + \underbrace{\frac{\partial F}{\partial \dot{x}} \delta \dot{x}} \right) dt = 0$$

ここで. $\int_{t_0}^{t_1} \left(\frac{\partial F}{\partial \dot{x}} \delta \dot{x} \right) dt = \int_{t_0}^{t_1} \left\{ \frac{\partial F}{\partial \dot{x}} \left(\frac{d}{dt} \delta x \right) \right\} dt$

$$= \underbrace{\left[\frac{\partial F}{\partial \dot{x}} \delta x \right]_{t_0}^{t_1}}_{0} - \int_{t_0}^{t_1} \left\{ \left(\frac{d}{dt} \frac{\partial F}{\partial \dot{x}} \right) \delta x \right\} dt$$

よって $\int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} \right) \delta x\, dt = 0$

$\Rightarrow \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) - \frac{\partial F}{\partial x} = 0$ Euler eq.

$L(q,\dot q) \equiv T - U$ とすると

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = 0$$ をみたす。 Euler-Lagrange

— なぜ $L = T - U$ と書けるか?

一般に、

$$p_i = \frac{\partial T}{\partial \dot x_i} \quad \text{or} \quad p_i = \frac{\partial T}{\partial \dot q_i} \qquad (\text{共役} : p, q)$$

$$= \frac{\partial T}{\partial \dot x_1}\frac{\partial \dot x_1}{\partial \dot q_i} + \frac{\partial T}{\partial \dot x_2}\frac{\partial \dot x_2}{\partial \dot q_i} + \cdots + \frac{\partial T}{\partial \dot x_{3N}}\frac{\partial \dot x_{3N}}{\partial \dot q_i}$$

$$= \sum_{j=1}^{3N} \frac{\partial T}{\partial \dot x_j} \underbrace{\frac{\partial \dot x_j}{\partial \dot q_i}}_{= \frac{\partial x_j}{\partial q_i}}$$

$$\leftarrow \frac{\partial \dot x_j}{\partial \dot q_i} = \frac{\partial x_j}{\partial q_i}$$ (要証明)

ビブン

$$\rightarrow \dot p_i = \underbrace{\sum_{j=1}^{3N} \underbrace{\frac{d}{dt}\Big(\underbrace{\frac{\partial T}{\partial \dot x_j}}_{m_j \dot x_j}\Big)}_{m_j \ddot x_j = F_j} \frac{\partial x_j}{\partial q_i}}_{\sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i} \equiv Q_i} + \underbrace{\sum_{j=1}^{3N} \frac{\partial T}{\partial \dot x_j}\left(\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right)\right)}_{\frac{\partial T}{\partial q_i}}$$

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right) = \frac{\partial \dot x_j}{\partial q_i}$$ (要証明)

Q_i: 一般力

よって $$\dot p_i = Q_i + \frac{\partial T}{\partial q_i}$$

$$\dot p_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right)$$

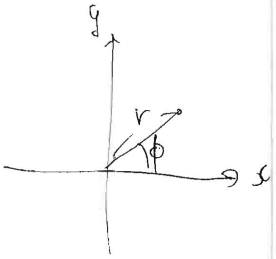
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} = Q_i = -\frac{\partial U}{\partial q_i}$$

$$\leftarrow Q_i \equiv -\frac{\partial U}{\partial q_i}$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial (T-U)}{\partial q_i} = 0$$

$L \equiv T - U$ として $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q_i} = 0$$

例. 2次元円筒座標



$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases} \rightarrow \begin{aligned} \dot{x} &= \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{y} &= \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{z} &= \dot{z} \end{aligned}$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\phi})^2 + \dot{z}^2\right)$$

$$U(x, y, z) = U(r\cos\phi, r\sin\phi, z)$$

$$L = T - U$$

Lagrange eq.

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{r}}}_{m\dot{r}}\Big) - \frac{\partial L}{\partial r} = 0 \qquad \frac{\partial L}{\partial r} = mr(\dot{\phi})^2 - \frac{\partial U}{\partial r}$$

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{\phi}}}_{mr^2\dot{\phi}}\Big) - \frac{\partial L}{\partial \phi} = 0 \qquad \frac{\partial L}{\partial \phi} = -\frac{\partial U}{\partial \phi}$$

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{z}}}_{m\dot{z}}\Big) - \frac{\partial L}{\partial z} = 0 \qquad \frac{\partial L}{\partial z} = -\frac{\partial U}{\partial z}$$

よって 運動eqs.

$$m\ddot{r} = mr(\dot{\phi})^2 - \frac{\partial U}{\partial r}$$

$$m\frac{d(r^2\dot{\phi})}{dt} = -\frac{\partial U}{\partial \phi}$$

$$m\ddot{z} = -\frac{\partial U}{\partial z}$$

ϕによらない $\longrightarrow \frac{d}{dt}(mr^2\dot{\phi}) = 0$

$r\dot{\phi}$: 一定

角運動量.

Legendre transform

$x, y, z, \ldots$ 独立変数 $\longrightarrow \Phi(x, y, z, \ldots)$

$$d\Phi = X\,dx + Y\,dy + Z\,dz + \cdots$$

$$\longrightarrow \frac{\partial \Phi}{\partial x} = X, \quad \frac{\partial \Phi}{\partial y} = Y, \quad \frac{\partial \Phi}{\partial z} = Z, \quad \cdots$$

変数 $x \rightarrow X$ へ変換

$$\Phi(x, y, z, \ldots) \rightarrow \Psi(X, y, z, \ldots) = \Phi(x, y, z, \ldots) - Xx.$$

このとき、

$$d\Psi = d(\Phi - Xx) = d\Phi - d(Xx)$$
$$= \cancel{d}\,(X\,dx + Y\,dy + Z\,dz + \cdots) - x\,dX - X\,dx$$
$$= -x\,dX + Y\,dy + Z\,dz + \cdots$$

よって、$x \rightarrow X$ へ変数が変わる。(Ledendre 変換)　"x と X は共役"

例. $dU = dQ - pdV$　また、$dQ = TdS$

$\rightarrow dU = TdS - pdV$　($U(S, V)$ と書ける)

$$U(S, V) \rightarrow H(S, p) \equiv U + pV$$

$$dH = \underbrace{dU}_{TdS - pdV} + Vdp + pdV = TdS + Vdp \longrightarrow H(S, p)$$

$$\rightarrow F(T, V) \equiv U - TS \qquad \because)\ dF = \overbrace{dU}^{TdS - pdV} - TdS - SdT = -SdT - pdV$$

$$G(p, T) \equiv U - TS + pV \qquad \because)\ dG = dU - TdS - SdT + Vdp + pdV = -SdT + Vdp$$

正準方程式.

$\dot{q}_i$に共役な $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

$$\frac{d}{dt}(p_i) - \frac{\partial L}{\partial q_i} = 0 \longrightarrow \frac{\partial L}{\partial q_i} = \dot{p}_i$$

Legendre変換　$H = \sum_i^{3N} p_i \dot{q}_i - L$　を行う.

$$\longrightarrow \left.\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i} = -\dot{p}_i \end{aligned}\right\}$$ 正準方程式.

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial (T-U)}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$$ (Uは$\dot{q}$によらない)

$$T = \sum_i \frac{1}{2} m_i \dot{x}_i^2 = \sum_i \frac{1}{2} m_i \left(\sum_{j=1} \frac{\partial x_i}{\partial q_j} \dot{q}_j \right)^2 \longleftarrow \dot{x}_i = \sum_{j=1} \frac{\partial x_i}{\partial q_j} \dot{q}_j$$

このとき. $\sum_i^{3N} p_i \dot{q}_i = \sum_i^{3N} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \dot{q}_i = 2T$

よって $H = 2T - L$
$= 2T - (T - U) = T + U$.

変分から正準方程式を導く

$$\delta I = \delta \int_{t_0}^{t_1} L(q, \dot{q})\, dt = 0$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$= \delta \int_{t_0}^{t_1} \left(\sum_i p_i \dot{q}_i - H \right) dt = 0$$

$$= \int_{t_0}^{t} \sum_i \left(\underbrace{(p_i \delta \dot{q}_i + \dot{q}_i \delta p_i)}_{= \underbrace{\left[p_i \delta q_i \right]_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1} \dot{p}_i \delta q_i \, dt} - \left(\frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i \right) \right) dt$$

$$\delta I = \int_{t_0}^{t_1} \sum_i \left(\left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right) dt = 0$$

そのための条件は、

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Poisson 括弧

物理量 $A(q_i, p_i, t)$ とする.

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i} \underbrace{\frac{\partial q_i}{\partial t}}_{\dot{q}_i} + \frac{\partial A}{\partial p_i} \underbrace{\frac{\partial p_i}{\partial t}}_{\dot{p}_i} \right)$$

$$= \frac{\partial A}{\partial t} + \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$= \frac{\partial A}{\partial t} + \{A, H\}$$ と書く.

t によらない. $\longrightarrow$ $\dot{q}_i = \{q_i, H\}$

$\dot{p}_i = \{p_i, H\}$ ハミルトン正準 eqs.

$\{q_i, q_j\} = 0$

$\{p_i, p_j\} = 0$

$\{q_i, p_j\} = \delta_{ij}$ $\qquad \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}$$ の証明 (2D極座標)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \longrightarrow \begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{cases}$$

$$\begin{cases} \dfrac{\partial x}{\partial r} = \cos\theta & \dfrac{\partial x}{\partial \theta} = -r\sin\theta \\ \dfrac{\partial y}{\partial r} = \sin\theta & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{cases}$$

$$\begin{cases} \dfrac{\partial \dot{x}}{\partial \dot{r}} = \cos\theta & \dfrac{\partial \dot{x}}{\partial \dot{\theta}} = -r\sin\theta \\ \dfrac{\partial \dot{y}}{\partial \dot{r}} = \sin\theta & \dfrac{\partial \dot{y}}{\partial \dot{\theta}} = r\cos\theta \end{cases}$$

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}$$

正準変換

$P = P(p, q, t) \quad Q = Q(p, q, t)$ としたとき、

$\dot{Q} = \dfrac{\partial K}{\partial P}, \quad \dot{P} = -\dfrac{\partial K}{\partial Q}$ のとき (Q, P)は Kに対する正準変換である。

$$\delta \int \left(p\dot{q} - H(q, p, t) \right) dt = \delta \int \left(P\dot{Q} - K(Q, P, t) \right) dt = 0$$

$$\Rightarrow \dot{q}p - H(q, p, t) = \dot{Q}P - K(Q, P, t) + \frac{dW}{dt}$$

$$\Rightarrow dW = p\,dq - P\,dQ + [K(Q, P, t) - H(q, p, t)]\,dt \quad ⊗$$

⊗は Wが qとQとtの関数　$W_1(q, Q, t)$

$$dW_1 = p\,dq - P\,dQ + [K(Q, P, t) - H(q, p, t)]\,dt$$

$$= p\,dq - d(PQ) + Q\,dP + [K - H]\,dt$$ $d(W_1 + PQ) = dW_2$

$$= d(pq) - q\,dp - P\,dQ + [K - H]\,dt$$ $d(W_1 - pq) = dW_3$

$$= d(pq - PQ) - q\,dp + Q\,dP + [K - H]\,dt$$ $d(W_1 - pq + PQ) = dW_4$

$p = \dfrac{\partial W_1(q, Q, t)}{\partial q}$ をQについて解く。

W_1 given として

・1式より $Q(q, p)$ を求める

・2式より $P(q, p)$ を求める

4つの関係式を得る

1. $W_1(q, Q, t)$ $\quad p = \dfrac{\partial W_1}{\partial q}, \quad P = -\dfrac{\partial W_1}{\partial Q}, \quad K = H + \dfrac{\partial W_1}{\partial t}$

2. $W_2(q, P, t) = W_1(q, Q, t) + PQ$

$$p = \frac{\partial W_2}{\partial q}, \quad Q = \frac{\partial W_2}{\partial P}, \quad K = H + \frac{\partial W_2}{\partial t}$$

3. $W_3(p, Q, t) = W_1(q, Q, t) - pq$

$$q = -\frac{\partial W_3}{\partial p}, \quad P = -\frac{\partial W_3}{\partial Q}, \quad K = H + \frac{\partial W_3}{\partial t}$$

4. $W_4(p, P, t) = W_1(q, Q, t) - pq + PQ$

$$q = -\frac{\partial W_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P}, \quad K = H + \frac{\partial W_4}{\partial t}$$

例. 一次元調和振動子 $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$ において

$(q, p) \to (Q, P)$ への変換.

1. $p = P$, $q = Q + aP$ について.

$$K(Q, P) = \frac{1}{2m}P^2 + \frac{1}{2}k(Q + aP)^2$$

$$\dot{Q} = \dot{q} - a\dot{P} = \frac{p}{m} - a\dot{p} = \frac{p}{m} + akq \quad \leftarrow \dot{p} = -kq \quad \frac{\partial H}{\partial q}$$

$$\frac{\partial K}{\partial P} = \frac{P}{m} + ak(Q + aP) = \frac{p}{m} + akq$$

よって $\dot{Q} = \frac{\partial K}{\partial P}$

また $\dot{P} = \dot{p} = -kq$, $-\frac{\partial K}{\partial Q} = -k(Q + aP) = -kq$

よって $\dot{P} = -\frac{\partial K}{\partial Q}$ 　正準変換となっている.

2. $p = \sqrt{m\omega}\, P$, $q = \frac{1}{\sqrt{m\omega}}Q$, $\omega^2 = \frac{k}{m}$ について.

$$K(Q, P) = \frac{1}{2}\omega(P^2 + Q^2) \qquad \dot{Q} = \sqrt{m\omega}\,\dot{q} = \sqrt{\frac{\omega}{m}}\,p \qquad \left(\dot{q} = \frac{p}{m}\right)$$

正準eqより

$$\frac{\partial K}{\partial P} = \omega P = \sqrt{\frac{\omega}{m}}\,p \qquad \text{よって } \dot{Q} = \frac{\partial K}{\partial P}$$

$$\dot{P} = \frac{1}{\sqrt{m\omega}}\dot{p} = \frac{1}{\sqrt{m\omega}}(-kq) \qquad -\frac{\partial K}{\partial Q} = -\omega Q = \frac{1}{\sqrt{m\omega}}(-kq)$$

$$\dot{P} = -\frac{\partial K}{\partial Q}$$

よって正準変換である.

位相空間では $\left(\sqrt{\frac{2K}{\omega}}\right)^2 = P^2 + Q^2$ の円となる.

例2. 一次元調和振動子.

・ $W(q, Q, t) = qQ$ のとき.

$$P = -\frac{\partial W}{\partial Q} = -q. \quad p = \frac{\partial W}{\partial q} = Q \text{ なので}$$

$$Q = p. \quad K = \frac{1}{2m}Q^2 + \frac{1}{2}kP^2$$

・ $W(q, Q, t) = \sqrt{mk}\, qQ$ のとき. $\rightarrow P = -\sqrt{mk}\, q$. $p = \sqrt{mk}\, Q$ より

$$Q = \frac{1}{\sqrt{mk}}p, \quad K = \frac{1}{2m}P^2 + \frac{1}{2}kQ^2.$$

例3. 電磁場 $H = \frac{1}{2m}(\mathbb{p} - e\mathbb{A})^2 + e\phi$ である

$(q, p) \longrightarrow (Q, P)$ への母関数 $W(q, \mathbb{P}, t) = q \cdot \mathbb{P} + ef(q, t)$

とする (W_2)

このとき $\mathbb{E} = -\nabla\phi - \frac{\partial \mathbb{A}}{\partial t}$, $\mathbb{B} = \nabla \times \mathbb{A}$ を不変に保つ

(証明) $\mathbb{p} = \nabla_q W = \mathbb{P} + e\nabla_q f(q, t)$ (a)

$$\mathbb{Q} = \nabla_{\mathbb{P}} W = \mathbb{q}$$

$$K(Q, P) = H(q, p) + \frac{\partial W}{\partial t} = \frac{1}{2m}(\mathbb{p} - e\mathbb{A})^2 + e\phi + e\frac{\partial f}{\partial t}$$

$$\overset{(a)}{=} \frac{1}{2m}\left(\mathbb{P} - e(\mathbb{A} - \nabla_q f)\right)^2 + e\left(\phi + \frac{\partial f}{\partial t}\right)$$

いま. $\mathbb{A} - \nabla_q f = \mathbb{A}'$, $\phi + \frac{\partial f}{\partial t} = \phi'$ とすると.

$$K(Q, P) = \frac{1}{2m}(\mathbb{P} - e\mathbb{A}')^2 + e\phi'$$

$\phi', \mathbb{A}'$ による $\mathbb{E}, \mathbb{B}$ を求めると

$$\mathbb{E}' = -\left(\nabla\phi' + \frac{\partial \mathbb{A}'}{\partial t}\right) = -\left(\nabla\phi + \frac{\partial \mathbb{A}}{\partial t}\right) = \mathbb{E}$$

$$\mathbb{B}' = \nabla \times \mathbb{A}' = \nabla \times (\mathbb{A} - \nabla f)$$

$$= \nabla \times \mathbb{A} - \nabla \times \nabla f = \nabla \times \mathbb{A} = \mathbb{B} \qquad (\nabla \times \nabla f = 0)$$

よって. $\mathbb{E}, \mathbb{B}$ は不変に保たれる変換となっている.

ハミルトン－ヤコビ方程式

Hamilton － Jacobi

$(q, p) \longrightarrow (Q, P)$

$$\dot{Q} = \frac{\partial K}{\partial P} \qquad \dot{P} = -\frac{\partial K}{\partial Q}$$

$$K = H + \frac{\partial W_2}{\partial t}$$

$$p = \frac{\partial W_2}{\partial q}$$

もし、$K \equiv 0$ となる母関数があれば Q と P は定数（運動の積分）

$$\frac{\partial W_2}{\partial t} + H\left(q_1, \cdots q_N, \frac{\partial W_2}{\partial q_1}, \cdots, \frac{\partial W_2}{\partial q_N}, t\right) = 0$$ となる。

解を S として、$\dfrac{\partial S}{\partial t} + H\left(q, \dfrac{\partial S}{\partial q}, t\right) = 0$

S は $q_1, \cdots, q_N$, t の $N+1$ コの変数から成る。

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\frac{dq}{dt} = -H + p\dot{q} = L \Rightarrow S = \int L\,dt$$

例、一次元 $H = \dfrac{p^2}{2m} + U(x)$ について、

HJ: $\dfrac{\partial S(x,t)}{\partial t} + H\left(x, \dfrac{\partial S}{\partial x}\right) = 0$ を解く。

エネルギー保存 $H = E$ より $\dfrac{\partial S(x,t)}{\partial t} = -E$

また、$\dfrac{\partial S(x,t)}{\partial x} = p = \sqrt{2m(E - U(x))}$

これより、$S(x,t) = -Et + \int dx \sqrt{2m(E-U(x))}$

自由粒子 $(U=0)$、定数 4ン

$$S(x,t) = -Et + x\sqrt{2mE} = -Et + px$$

Hamilton-Jacobi eq から Schrödinger eq を導く.

自由粒子の平面波 $\psi(x,t) = e^{i(kx-\omega t)} = e^{i(px-Et)/\hbar} = e^{iS(x,t)/\hbar}$ Ⓐ

$p = \hbar k$, $E = \hbar\omega$　　$S = -Et + px$

逆に. $S(x,t) = \dfrac{\hbar}{i} \ln \psi(x,t)$ となる.

$$\frac{\partial S}{\partial t} = -\frac{i\hbar}{\psi}\frac{\partial \psi}{\partial t}$$

$$H\left(x, \frac{\partial S}{\partial x}\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U(x) = -\frac{\hbar^2}{2m}\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial x}\right)^2 + U(x).$$

これを HJ eq $\dfrac{\partial S}{\partial t} + H\left(x, \dfrac{\partial S}{\partial x}\right) = 0$ に代入して.

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\underbrace{\frac{1}{\psi}\left(\frac{\partial \psi}{\partial x}\right)^2}_{\psi\text{に関して線型でない.}} + U(x)\psi$$ となる.

Ⓐより.

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{i}{\hbar}\frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x} e^{i\frac{S}{\hbar}}\right) = \frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2} e^{i\frac{S}{\hbar}} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2 e^{i\frac{S}{\hbar}}$$

$$= \left[\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2\right]\psi$$ となる.

$\left|\dfrac{\partial^2 S}{\partial x^2}\right| \ll \dfrac{1}{\hbar}\left|\dfrac{\partial S}{\partial x}\right|^2$ であれば $\dfrac{\partial^2 S}{\partial x^2} \sim \dfrac{1}{\psi}\left(\dfrac{\partial \psi}{\partial x}\right)^2$ とできる

$$\Rightarrow\ i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U(x)\psi$$ となる.

コメント
remark　$\dfrac{1}{\hbar}\dfrac{\Delta S}{(\Delta x)^2} \ll \dfrac{1}{\hbar^2}\dfrac{(\Delta S)^2}{(\Delta x)^2} \Rightarrow \hbar \ll \Delta S$

ΔS が $\hbar$ より十分大きい古典論で成り立つ??　変

$\Delta S \sim \hbar$ であるべき (量子論)

量子論 $\xrightarrow{\hbar \to 0}$ 古典論　　一端にすぎない.

ネーターの定理

一般座標 q について、パラメータ s として $q(s)$, $\dot{q}(s)$ とする。$L(q,\dot{q})$ が不変なためには…

$$\frac{d}{ds}L(q(s),\dot{q}(s)) = \frac{\partial L(q,\dot{q})}{\partial q}\frac{\partial q}{\partial s} + \frac{\partial L(q,\dot{q})}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial s} = 0$$

$s\to 0$ として、

$$\begin{aligned} 0 &= \frac{\partial L(q,\dot{q})}{\partial q}\frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial s} \\ &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\frac{\partial q}{\partial s} \\ &= \frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial s}}\right) \end{aligned}$$

$$\longrightarrow I = \sum_{k=1}^{K}\frac{\partial L(q,\dot{q})}{\partial \dot{q}_k}\frac{\partial q_k}{\partial s}$$ が保存する

ネーターカレント。

例、$L=\sum \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ について、

1. $x_a(s)=x_a+s$ として　$(a=1,\cdots,N)$

$$I_x = \sum_{a=1}^{N}\underbrace{\frac{\partial L}{\partial \dot{x}_a}}_{m\dot{x}}\underbrace{\frac{\partial x_a(s)}{\partial s}}_{1}\Bigg|_{s\to 0} = \sum_{a=1}^{N} m_a\dot{x}_a$$ 運動量保存則

2. 円筒座標　$x(s)=r\cos(\theta+s)$, $y(s)=r\sin(\theta+s)$

$$I = \sum\left(\frac{\partial L}{\partial \dot{x}}\frac{\partial x(s)}{\partial s} + \frac{\partial L}{\partial \dot{y}}\frac{\partial y(s)}{\partial s}\right) = \sum m(-\dot{x}y+\dot{y}x)$$

角運動量保存則

3. 時間並進

3. 時間発展

$t \longrightarrow \tau(t)$ と変換する

$$\delta S = \delta \int_{t_A}^{t_B} L\left(q, \frac{dq}{dt}, t\right) = \delta \int_{\tau_A}^{\tau_B} L\left(q, \frac{dq/d\tau}{dt/d\tau}, t(\tau)\right) \frac{dt}{d\tau} d\tau = 0$$

t と $\frac{dt}{d\tau}$ を独立変数として

$$L'\left(q, \frac{dq}{d\tau}, t, \frac{dt}{d\tau}\right) = L\left(q, \frac{dq/d\tau}{dt/d\tau}, t(\tau)\right) \frac{dt}{d\tau}$$

このとき、t に関する Lagrange eq は、

$$\underbrace{\frac{d}{d\tau}\left(\frac{\partial L'}{\partial\left(\frac{dt}{d\tau}\right)}\right)} - \frac{\partial L'}{\partial t} = 0$$

$$\longrightarrow \frac{\partial L'}{\partial\left(\frac{dt}{d\tau}\right)} = L - \frac{\frac{dq}{d\tau}}{\left(\frac{dt}{d\tau}\right)^2} \cdot \frac{\partial L}{\partial\left(\frac{dq}{dt}\right)} \frac{dt}{d\tau}$$

$$= L - \frac{\frac{dq}{d\tau}}{\frac{dq}{dt}} \cdot p = L - p\dot{q} = -H$$

一方、$\frac{\partial L'}{\partial t} = \frac{\partial L}{\partial t} \frac{dt}{\partial \tau} = \frac{\partial(p\dot{q} - H)}{\partial t} \frac{dt}{d\tau} = -\frac{\partial H}{\partial t} \cdot \frac{dt}{d\tau}$

よって、$\frac{d(-H)}{d\tau} + \frac{\partial H}{\partial t} \frac{dt}{d\tau} = 0 \longrightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t}$

H が陽に t を含まなければ、$\frac{\partial H}{\partial t} = 0$ なので

$\frac{dH}{dt} = 0$ となる。

これは、時間発展に対するエネルギー保存則を意味している